

BLACK HOLE QUBIT CORRESPONDENCE FROM QUANTUM CIRCUITS

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We consider the black hole qubit correspondence (BHQC) from quantum circuits, taking into account the use of gate operations with base in the formulation of wrapped brane qubits. We interpret these quantum circuits with base on the BHQC classification of entanglement classes and apply in specific examples as the generation of Bell, GHZ states, quantum circuit teleportation and consider the implementation of interchanges in SUSY, black hole configurations, Freudenthal and rank system constructions. These results are discussed from the superstring viewpoint showing that the importance of the construction of the physical states formed by the entanglement of geometrical entities by cohomological operations automatically allows the preservation of different amounts of SUSY in the compactification process given an alternative to the case when fluxes are introduced in the game: the generalized Calabi-Yau of Hitchin.

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1. Introduction

One of the crucial ingredients to the scenario of extra dimensions is a brane on which particles of standard model are localized. In string theory, fields can naturally be localized on D-branes due their open string endings¹. Although, extra dimensions

are generally proposed in compactified form, the localization mechanism for gravity can lead to new possibilities². Due to gravitational interactions between brane in uncompactified 5D-space, a four dimensional Newtonian behavior can be achieved when the bulk cosmological constant and the brane tension are related. Additionally, gravitons can be localized on branes by separating two patches of AdS5 space-time³. Static 4D-brane universes can exist under the requirement that their tension is fine-tuned with the bulk cosmological constant. Interesting models in which extra dimensions can also take advantage in the AdS/CFT correspondence, where strongly coupled 4D theory to 5D warped dimensions can be related⁴. Extensions to charged branes and thermal strings are also possible scenarios^{6,5}. A more recent fact is the string-theoretic interpretation of the black holes in terms of Dp-branes wrapping around six compactified dimensions associated to qubits from quantum information (QI)⁷. This is the so-called black hole qubit correspondence (BHQC)⁸. It has lead to important achievements as the association between black hole entropy emerging from the solution of $N = 2$ supergravity STU model of string compactification and tripartite entanglement measurement^{9,10}, the association between the black hole configurations in STU supergravity and entanglement state classification^{7,11} and the identification of the Hilbert space of the qubits associated to the wrapped branes inside the cohomology of the extra dimensions¹². In fact, many important results were obtained^{14,16,13,17,15,19,18,20} (see²¹ for a more complete review). It is believed that this BHQC can also be extended to the context of supergeometries^{26,27,28,29}.

In this paper, we propose a BHQC from the point of view of quantum circuits and give the corresponding interpretation. Although recently the role of entanglement and superpositions were explored in BHQC, quantum circuits was not explored clearly in this context, mainly the interpretation in the string side of the BHQC. As we will discuss, this step is fundamental to explore the BHQC in terms of quantum circuits. We explore the correspondence and its association to quantum information, giving a clear interpretation for complete correspondence with BHQC, moving steps forward to the role of quantum circuits in the BHQC, in particular to explore string theoretical scenarios. In agreement with previous proposals, we first associate the wrapped brane qubits in the BHQC according to an one-to-one association⁷ and then build the necessary gate operations to implement quantum circuits^{24,25}. As instances, we obtain specific quantum circuits as the generation of Bell states, quantum teleportation circuit^{22,23} and generation of GHZ states. The fundamental point we propose the interpretation of these quantum circuits with base on the BHQC for the classification of entanglement classes^{7,11}. Although this clear structure is dressed in a quantum information formalism, the presence of the interpretation gives the key point for establishing the BHQC in this context. This step was not clarified in the previous proposals.

We organized the paper as follows: In Sec. II, we consider gate operations with base on the BHQC of wrapped brane qubits and perform quantum circuits in BHQC. In Sec. III, we address the BHQC of quantum circuits in the interpreting these

systems from the STU black hole side. Section IV is devoted to make some brief remark concerning the superstring viewpoint of the results. Finally, our conclusions are reserved to Sec.V.

2. BHQC for qubits from extra dimensions, generation of Bell states and quantum teleportation

The relations in the BHQC for qubits from extra dimensions start with the associations between one-forms and one-mode states $\Omega \leftrightarrow |0\rangle, \bar{\Omega} \leftrightarrow |1\rangle$, where the vacuum state is can be associated to an holomorphic three-form in the Calabi-Yau space and K is the Kähler potential¹². The orthonormality relations are written as

$$\int_{T^2} \Omega \wedge * \bar{\Omega} \leftrightarrow \langle 0|0\rangle = 1, \quad (1)$$

$$\int_{T^2} \bar{\Omega} \wedge * \Omega \leftrightarrow \langle 1|1\rangle = 1, \quad (2)$$

$$\int_{T^2} \bar{\Omega} \wedge * \bar{\Omega} \leftrightarrow \langle 1|0\rangle = 0, \quad (3)$$

$$\int_{T^2} \Omega \wedge * \Omega \leftrightarrow \langle 0|1\rangle = 0. \quad (4)$$

The action of the Hodge star operator $*$, that introduces a phase term on $|0\rangle$, reads $*|0\rangle = -|0\rangle, *|1\rangle = |1\rangle$. In a superposed state, $*(|1\rangle \pm |0\rangle) = |1\rangle \mp |0\rangle$. On the other hand, the action of the flat Kähler covariant derivative $D_{\hat{\tau}}\Omega = (\bar{z}^{\tau} - z^{\tau})(\partial_{\tau} + \frac{1}{2}\partial_{\tau}K)\Omega$ follow the rules $D_{\hat{\tau}}\Omega = \bar{\Omega}, D_{\hat{\tau}}\bar{\Omega} = 0$, and the flat adjoint covariant derivatives follows the correspondences $D_{\hat{\tau}}\Omega = 0$ and $D_{\hat{\tau}}\bar{\Omega} = \Omega$, leading to BHQC with bit-flippers $D_{\hat{\tau}}\Omega \leftrightarrow \uparrow |0\rangle = |1\rangle, D_{\hat{\tau}}\bar{\Omega} \leftrightarrow \uparrow |1\rangle = 0, D_{\hat{\tau}}\Omega \leftrightarrow \downarrow |0\rangle = 0, D_{\hat{\tau}}\bar{\Omega} \leftrightarrow \downarrow |1\rangle = |0\rangle$. A general qubit state in extra dimensions can then be represented by a non-normalized qubit $|\Gamma\rangle = \alpha|1\rangle + \beta|0\rangle$. The basic elements $*, \uparrow, \downarrow, |1\rangle, |0\rangle$ can be used to implement the BHQC to a large range of combinations. The Hodge star operators and the covariant derivatives can be combined to define operators

$$\lambda_1 = * \uparrow + \uparrow *, \quad (5)$$

$$\lambda_2 = * \downarrow + \downarrow *, \quad (6)$$

$$\lambda_3 = * \downarrow + \uparrow *, \quad (7)$$

$$\lambda_4 = * \uparrow + \downarrow *. \quad (8)$$

These operators act on one-mode states $|0\rangle$ and $|1\rangle$ leading to the following relations $\lambda_1|j\rangle = 0, \lambda_2|j\rangle = 0, \lambda_3|j\rangle = -|j \oplus 1\rangle, \lambda_4|j\rangle = |j \oplus 1\rangle$, where $j = 0, 1 \in \mathbb{Z}_2$. An immediate consequence of this operation is $\lambda_3^2 \leftrightarrow I, \lambda_4^2 \leftrightarrow I$. The action of these operators on qubit states are $\lambda_3(\alpha|0\rangle + \beta|1\rangle) = -(\alpha|1\rangle + \beta|0\rangle)$ and $\lambda_4(\alpha|0\rangle + \beta|1\rangle) = (\alpha|1\rangle + \beta|0\rangle)$. These operators are then equivalent to a NOT gate²⁵ and are related to each other by means of $\lambda_4 = -\lambda_3$. It follows, all the one-mode gate operations can be realized by combinations of the actions of the operator λ_4

and its square $\lambda_4^2 = I$. We can also have Hadamard gates by means of the operations $(I + *\lambda_4)|1\rangle = |1\rangle - |0\rangle$ and $(I + *\lambda_4)|0\rangle = |0\rangle + |1\rangle$ or $\lambda_4(I + *\lambda_4)|1\rangle = |0\rangle - |1\rangle$ and $\lambda_4(I + *\lambda_4)|0\rangle = |0\rangle + |1\rangle$. A σ_2 -type gate can be obtained from $\lambda_4*|0\rangle = -|1\rangle$ and $\lambda_4*|1\rangle = |0\rangle$ or $\lambda_3*|0\rangle = |1\rangle$ and $\lambda_3*|1\rangle = -|0\rangle$. In similar fashion, other gate operations can be built from suitable applications, leading to a BHQC for a universal quantum computation.

The correspondence can be generalized to n -mode case. In the case of a two-mode space, whose basis is $\{|00\rangle, |10\rangle, |01\rangle, |11\rangle\}$, two mode gate operation can be obtained from one-mode ones. For instance,

$$* \otimes * |ij\rangle = \begin{cases} |ii\rangle, & \text{if } i = j; \\ -|ij\rangle, & \text{otherwise.} \end{cases} \quad (9)$$

$$\uparrow \otimes \uparrow |ij\rangle = \begin{cases} |11\rangle, & \text{if } i = j = 0; \\ 0, & \text{otherwise.} \end{cases} \quad (10)$$

$$\downarrow \otimes \downarrow |ij\rangle = \begin{cases} |00\rangle, & \text{if } i = j = 1; \\ 0, & \text{otherwise.} \end{cases} \quad (11)$$

It is also easy to check that $\uparrow \otimes \downarrow |01\rangle = |10\rangle$ and $\downarrow \otimes \uparrow |10\rangle = |01\rangle$, zero otherwise. As in the one-mode case, we can define new operators

$$\Lambda_1 = * \otimes \uparrow + \uparrow \otimes *, \quad (12)$$

$$\Lambda_2 = * \otimes \downarrow + \downarrow \otimes *, \quad (13)$$

$$\Lambda_3 = * \otimes \uparrow + \downarrow \otimes *, \quad (14)$$

$$\Lambda_4 = * \otimes \downarrow + \uparrow \otimes *, \quad (15)$$

that lead to the following results $\Lambda_1|00\rangle = -(|01\rangle + |10\rangle)$, $\Lambda_1|11\rangle = 0$, $\Lambda_1|01\rangle = |11\rangle$, $\Lambda_1|10\rangle = |11\rangle$, $\Lambda_2|00\rangle = 0$, $\Lambda_2|11\rangle = |01\rangle + |10\rangle$, $\Lambda_2|01\rangle = -|00\rangle$, $\Lambda_2|10\rangle = -|00\rangle$, $\Lambda_3|00\rangle = -|01\rangle$, $\Lambda_3|11\rangle = |01\rangle$, $\Lambda_3|01\rangle = 0$, $\Lambda_3|10\rangle = |11\rangle - |00\rangle$, $\Lambda_4|00\rangle = -|01\rangle$, $\Lambda_4|11\rangle = |01\rangle$, $\Lambda_4|01\rangle = 0$, $\Lambda_4|10\rangle = |11\rangle - |00\rangle$. In particular, we can verify $\Lambda_2\Lambda_1|00\rangle = 2|00\rangle$, $\Lambda_1\Lambda_2|11\rangle = 2|11\rangle$.

A controlled not (CNOT) gate can be implemented considering the conjugation rules

$$\Omega \otimes \Omega \rightarrow \Omega \otimes \Omega, \quad (16)$$

$$\Omega \otimes \bar{\Omega} \rightarrow \Omega \otimes \bar{\Omega}, \quad (17)$$

$$\bar{\Omega} \otimes \Omega \rightarrow \bar{\Omega} \otimes \bar{\Omega}, \quad (18)$$

$$\bar{\Omega} \otimes \bar{\Omega} \rightarrow \bar{\Omega} \otimes \Omega, \quad (19)$$

These rules corresponds to the operation

$$|ij\rangle \rightarrow |i\rangle|i \oplus j\rangle, \quad (20)$$

and then implement a CNOT gate, where i corresponds to the control and j to the target. As can be verified, this gate can be written as the action of the following operator $U_{CNOT}(i) = (I \otimes \lambda_4)^i$.

As direct application of the BHQC, we can verify the usual quantum circuits implementing Bell states and quantum teleportation. Taking the input state $|00\rangle$, we can apply a set of gate operations in order to generate all the Bell states. We apply the gate operation $(\ast \otimes \ast)\Lambda_1$ on the input state we have the first Bell state

$$|B_1\rangle = (\ast \otimes \ast)\Lambda_1|00\rangle = |01\rangle + |10\rangle, \quad (21)$$

The second Bell state $|B_2\rangle$ can be generated by applying the $(I \otimes \ast)$ gate operation

$$|B_2\rangle = (I \otimes \ast)|B_1\rangle = |01\rangle - |10\rangle. \quad (22)$$

The application of $(I \otimes \uparrow)$ leads to

$$|B_3\rangle = (I \otimes \uparrow)|B_2\rangle = |00\rangle - |11\rangle. \quad (23)$$

Next, applying $(I \otimes \ast)$, results

$$|B_4\rangle = (I \otimes \lambda_4\lambda_3)|B_3\rangle = |00\rangle + |11\rangle. \quad (24)$$

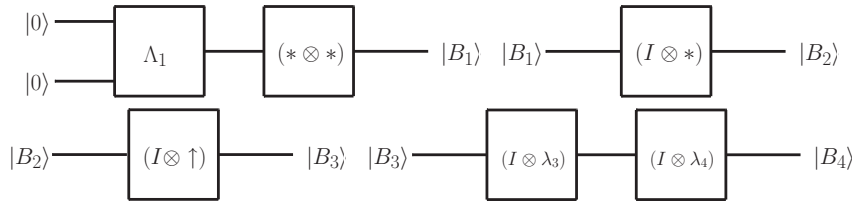


Fig. 1. (Color online) Quantum circuit for the generation of the bell states from qubits.

Let us consider now a qubit state with coefficients α and β unknown. We can realize a quantum state teleportation by means of a circuit teleportation²³ with the use of the previous gate operations. The initial state can be represented by

$$|\Gamma\rangle_a |B_4\rangle_{b_1 b_2}, \quad (25)$$

where b_1 correspond to the first mode and b_2 to the second mode of the Bell state. The coefficients of $|\Gamma\rangle_a = \alpha|0\rangle_a + \beta|1\rangle_b$ are generally unknown.

Under a CNOT gate where the qubed mode a is the control state, the state (25) is modified to

$$(\alpha|00\rangle_{ab_1} + \beta|11\rangle_{ab_1})|0\rangle_{b_2} + (\alpha|01\rangle_{ab_1} + \beta|10\rangle_{ab_1})|1\rangle_{b_2}. \quad (26)$$

Applying the Hadamard gate operation in the mode a , we arrive at

$$\begin{aligned} & [\alpha(|0\rangle_a + |1\rangle_a)|0\rangle_{b_1} + \beta(|1\rangle_a - |0\rangle_a)|1\rangle_{b_1}]|0\rangle_{b_2} \\ & + [\alpha(|0\rangle_a + |1\rangle_a)|1\rangle_{b_1} + \beta(|1\rangle_a - |0\rangle_a)|0\rangle_{b_1}]|1\rangle_{b_2}, \end{aligned} \quad (27)$$

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and applying a λ_4 operation on the mode a we arrive in the opposed Hadamard operation

$$\begin{aligned} & [\alpha(|0\rangle_a + |1\rangle_a)|0\rangle_{b_1} + \beta(|0\rangle_a - |1\rangle_a)|1\rangle_{b_1}]|0\rangle_{b_2} \\ & + [\alpha(|0\rangle_a + |1\rangle_a)|1\rangle_{b_1} + \beta(|0\rangle_a - |1\rangle_a)|0\rangle_{b_1}]|1\rangle_{b_2}. \end{aligned} \quad (28)$$

Since the usual projection relations applies to the one-forms, we can project the total state into the state $|00\rangle_{ab_1}$, given by the application of the projector

$$P_{ab_1}^{(0)} = (|00\rangle\langle 00|)_{ab_1},$$

on the whole state (28), the mode b_2 then assumes the state

$$\alpha|0\rangle_{b_2} + \beta|1\rangle_{b_2},$$

which corresponds to the quantum state teleportation of the qubed from the mode a to b_2 (figure 2).

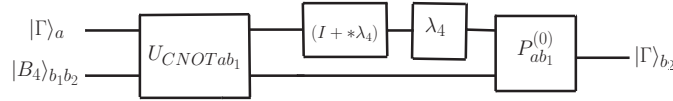


Fig. 2. (Color online) Sequence of quantum gate operations in the quantum circuit teleportation for qubeds.

3. Quantum circuits: BHQC interpretation

The association between the entropy of an STU black hole supergravity and a 3-tangle of a given tripartite state in one-to-one correspondence⁷ leads to a clear association with a purely tripartite entangled state, the GHZ state⁹. Considering a three qubit quantum circuit, this state is generated from a Bell state $|B_4\rangle_{a_1 a_2}$ and a third state $|0\rangle_b$, applying a CNOT gate in $a_2 b$, where a_2 is the control, we generate a GHZ state (figure 3). Alternatively, the same result is obtained if the

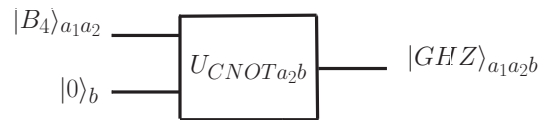


Fig. 3. (Color online) Quantum circuit generating a GHZ state.

control qubit is a_1 . We can just write, $k = 1, 2$,

$$U_{CNOT_{a_k b}}|B_4\rangle_{a_1 a_2} \otimes |0\rangle_b = |000\rangle_{a_1 a_2 b} + |111\rangle_{a_1 a_2 b}.$$

This state corresponds to four D3-branes intersecting over a string in the BHQC, according to the classification of three-qubit states as supersymmetric black holes ⁷. In this classification, a BHQC of quantum circuit can be used to generate different wrapping configurations of these intersecting D3-branes by the action of appropriate operators. In fact, if we consider a more general state corresponding to a STU black hole $|ABC\rangle = \sum_{ijk=0}^1 \alpha_{ijk} |ijk\rangle$, in particular, the triality interchanges and class changes can be implemented using BHQC of quantum circuits. In particular, the classes $A - B - C$, $A - BC$, W and GHZ as described in ⁷ can be interconnected by gate operations. We can implement for instance the an interchange of $A - B - C$ to $A - BC$ by applying a Hadamard gate in the third qubit and then a CNOT gate operations between the second and the third as control

$$|000\rangle \rightarrow |000\rangle + |001\rangle \rightarrow |101\rangle + |110\rangle.$$

This change corresponds to modify the SUSY configuration of the small black hole from 1/2 preserved to 1/4 preserved (figure 4). On the other hand, the quantum circuit for the generation of a GHZ state corresponds to a passage from a small (non-attractor) to a large (attractor) black hole with SUSY 1/8 preserved or completely broken (figure 5).

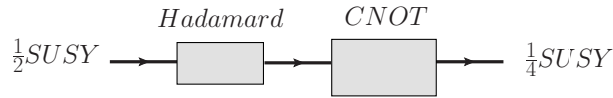


Fig. 4. (Color online) Change in the SUSY configuration of a small black hole corresponding to a BHQC quantum circuit.



Fig. 5. (Color online) Change from a small to a large black hole from a BHQC quantum circuit.

An implementation of STU black holes associated to four qubit systems ^{11,14} is made in analogous way to the previous circuits, we can derive a quantum circuit for four entanglement to implement a BHQC applying adequate quantum gate operations in the presence of an auxiliary qubit, for example, $|GHZ\rangle \otimes |0\rangle$ under a CNOT gate. The extension to quantum circuits with a higher number of qubits can be made straightforwardly by the introduction of new auxiliary qubits. This procedure corresponds to deal with a given n -form in the cohomology class of wrapped Dp -branes. Other mechanisms, as the process of moduli stabilization at the horizon associated to entanglement distillation to a GHZ state ¹⁵ can also be implemented in a more clear form from the perspective of BHQC in quantum circuits. Because the automorphism group corresponds to the U-duality group of a variety 4-dimensional supergravities, in the context of the Freudenthal's construction ³⁰, the BHQC can

be implemented to start from a three-qubit separable projective coset and implement the gate operations to move to a biseparable projective coset or two entangled qubits coset. For instance,

$$\begin{aligned} SL(2, C) \times SL(2, C) \times SL(2, C) &\rightarrow SL(2, C) \times SL(4, C) \\ &\rightarrow SL(6, C). \end{aligned}$$

As a consequence, quantum circuits can also be implemented to connect automorphism (SLOCC) groups in BHQC. In a FTS rank system, it corresponds to move from given rank system and other rank system, as, for example, starting from a rank 1 system and generate a rank 4, rank 3 system or rank 2a, 2b, 2c system. It is important to remind in these cases that, the conventional concept of matrix rank may be generalised to Freudenthal triple systems^{31,32} in a natural and authomorphic invariant manner.

4. Superstring viewpoint

As is well known Calabi-Yau manifolds are important in superstring theory because the ten conjectural dimensions are supposed to come as four of which we are aware, carrying some kind of fibration with fiber dimension six, and they leave some of the original supersymmetry (SUSY) unbroken^{33,34,35}. The importance of the states constructed via entanglement in the different quantum-information theoretical processes described here, is that as the starting point, the geometry of a Calabi-Yau manifold is used to define such states. Consequently, the topology of the Calabi-Yau manifold changes (not the dimension) making that the preserved SUSY under the compactification process also change. That means that we have a mechanism to control the preserved susy under the compactification process.

In resume: the importance of the construction of the physical states formed by the entanglement of geometrical entities by cohomological operations automatically allows the preservation of different amounts of SUSY in the compactification process given an alternative to the case when fluxes are introduced in the game: the generalized Calabi-Yau of Hitchin^{37,38}.

5. Conclusions

We have considered quantum circuits implemented in the context of BHQC with qubits from wrapped branes. Applying first to obtain quantum circuits to generation of Bell, GHZ states and teleportation, we then connect the states involved interpreting these quantum circuits in terms of the entanglement classes classification associated to the entropy and SUSY configurations of STU black holes⁷. As a consequence, we used our formulation to consider the interchange of SUSY, black hole configurations, Freudenthal's and rank system contructions by means of BHQC quantum circuits.

As we have clearly seen, the results show that the cohomological operations performed and proposed in this paper allow the amount of supersymmetry preserved,

be more flexible under compactification that from the quantum field theoretical point of view is extremely important due that the interplay between the SUSY preserved, the moduli space and the geometry/topology of the remanent (super) phase space

This proposal is also useful in the mechanism of moduli stabilization at the horizon^{15,36}, by considering corresponding BHQC quantum circuits, what can be implemented in a future work elsewhere.

In the context of Riemannian superspaces of^{28,29}, it is possible to reformulate consistently this construction at the operator level avoiding the black hole interpretation due that, as we have been shown, the black/hole entropy argument is not necessary because the spacetime carry itself the quantum/statistical properties (there is not dependence of such properties on a particular solution, as in the black hole case). This is also important in the construction of the physical states formed by the entanglement of geometrical entities by cohomological operations.

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